Motion Trajectory of a Ball in a 3 Dimensional Parabolic Disk

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Abstract: This study presents how to trace the motion of a ball in Mathematics when it is thrown in a three-dimensional parabolic space. Considering only the gravity acting on the ball, the free-fall position function is written as a vector valued function which takes the time variable t as an input and returns an output vector $\langle p_x(t), p_y(t), p_z(t) \rangle$. The contact event of the ball with the wall enforces to changes the moving direction of it based on the reflection theory of light, and the returned speed depends on the Coefficients of Restitution (COR) of the ball and wall. All sequential procedures are mathematically analyzed step by step, and the resultant piecewise position function is simulated/animated in MATLAB. With various CORs of the ball, we confirm that the increase of COR results in the increase of jumps in it.
Keywords: Projectile motion, Simulation of a ball, 3 dimensional parabolic disk

I. Introduction

As an effective teaching and learning methodology, project-based learning (PBL) has been advocated in much literature in many academic fields over the past decades.   To summarize, the positive effects of PBL are described that it deepens students' academic conceptual understandings and encourages them to do critical thinking through its applications [1]. Such positive effects of PBL are proved in educational levels ranging from K-12 education [2], [3], [4], [5] through undergraduate courses [6], [7], [8], [9].

As an extensive learning approach of PBL, the summer-undergraduate-research-experience (SURE) has proven to be a higher-impact educational practice for achieving academic excellence and gaining positive attitudes towards future career pursuits. Research on SURE has revealed an extensive array of professional benefits, such as research skills, and personal benefits, such as independence of intrinsic motivation for learning, the increase of self-confidence, and more advantage in any career path in [10], [11]. Many educators, particularly in the science-technology-engineering-mathematics (STEM) fields believe that a research experience also compels the students' interest in science careers [1]. Such a sequence of effects are linked to long-standing heightened performance in graduate degrees [12], [13], [14].

Given such information, federal funding agencies including the National Science Foundation (NSF) have supported undergraduate research by sponsoring professional development opportunities across the science and engineering disciplines [1].

This article introduces a sequential undergraduate research processes with the following factors:
- How it can be analyzed in Mathematics
- How it can be simulated in MATLAB with various CORs
- How it can be animated in MATLAB
- How the conclusion is drawn

through a summer-undergraduate-research-training (SURT), which is conducted by an undergraduate student who has just completed the only Calculus I in Mathematics and a general Physics course:

II. Question and Assumption

Question: When a ball is thrown in a 3 dimensional parabolic dish, how does the ball move and how does the coefficient of restitution (COR) of the ball work?

Assumptions:
a. The initial position and velocity of the ball are $< p_{x_0}, p_{y_0}, p_{z_0} >$ and $< v_{x_0}, v_{y_0}, v_{z_0} >$, respectively in a standard rectangular coordinate system
b.   The three dimensional parabolic dish is defined by $f(x, y, z) = ax^2 + by^2$ $(a > 0, b > 0)$.
c. The motion of the bouncing ball obeys projectile motion (or freefall motion)
d. The ball does not have any volume, like a particle but a mass

e. The ball reflects like a light when it contacts with an obstacle
f. The ball has a certain COR, α

There are certain types of forces acting on the ball, including gravity, drag force, the Magnus effect, the buoyancy. Gravity on an object is the downward force that attracts a body toward the center of the earth, or toward other physical bodies having mass. On the surface of Earth, it is defined by $\mathbf{g} \approx 9.8 \text{ m/sec}^2$. Drag force on an object is the force that resists its motion through a fluid, so-called air resistance. The Magnus effect is the generated side force on a spinning cylindrical or spherical solid immersed in a fluid. Buoyancy is the force exerted due to difference of the density between the object and fluid when the object is partially immersed or submerged. However, projectile motion is a form of motion of an object which has gravity as the only force of significance that acts on it. That is, the effects of air resistance are assumed to be negligible. Moreover, the ball maintains the horizontal velocity component without any other external horizontal force because of its inertia. See the book [15].

III. Mathematical Analysis

Once the ball hits the wall, it starts a new projectile motion with a new position vector and a new velocity vector until it hits the wall again. And thus we explain these sequential movements in the flow of below diagram:
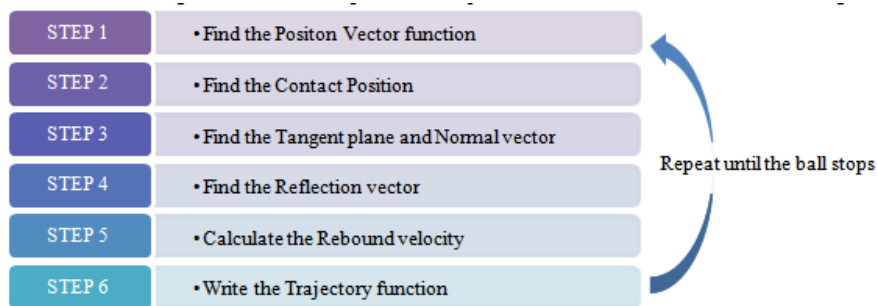
Step 1: Positon Vector function
In the absence of any other forces except a constant gravitational force $\mathbf{g}$ (almost 9.8 m/sec^2) acting on the ball, a set of position functions along the x-, y- and z- axes is given as

$$\begin{cases} x(t) = v_{x_0} t + p_{x_0} \\ y(t) = v_{y_0} t + p_{y_0} \\ z(t) = -\frac{1}{2}\mathbf{g}t^2 + v_{z_0} t + p_{z_0} \end{cases} \tag{1}$$

or as a vector valued function by

$$\mathbf{r}(t) = \; < v_{x_0} t + p_{x_0}, \qquad v_{y_0} t + p_{y_0}, \quad -\frac{1}{2}\mathbf{g}t^2 + v_{z_0} t + p_{z_0} > \tag{2}$$

wherethe independent variable t represents the time in second units. Here each vector component is denoted in the meter units.

Step 2: Contact Position
The first contact point is obtained at t_0 in which the position of the ball meets the parabolic wall. In mathematics the set of functions Eq. (1) satisfies the wall equation

$$z = ax^2 + by^2 \tag{3}$$

That is,

$$-\frac{1}{2}\mathbf{g}t^2 + v_{z_0}t + p_{z_0} = a\left(v_{x_0}t + p_{x_0}\right)^2 + b\left(v_{y_0}t + p_{y_0}\right)^2$$

This equation is then expanded and rearranged in the descending order by

$$\left(av_{x_0}^2 + bv_{y_0}^2 + \frac{1}{2}\mathbf{g}\right)t^2 + \left(2av_{x_0}p_{x_0} + 2bv_{y_0}p_{y_0} - v_{z_0}\right)t + \left(ap_{x_0}^2 + bp_{y_0}^2 - p_{z_0}\right) = 0$$

in a quadratic equation in terms of t. Using the quadratic formula to solve it, we have

$$t = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

where

$$A := av_{x_0}^2 + bv_{y_0}^2 + \frac{1}{2}\boldsymbol{g}$$
$$B := 2av_{x_0}p_{x_0} + 2bv_{y_0}p_{y_0} - v_{z_0}$$
$$C := ap_{x_0}^2 + bp_{y_0}^2 - p_{z_0}$$

Since $A > 0$ and $\sqrt{B^2 - 4AC} \geq 0$,

$$\frac{-B + \sqrt{B^2 - 4AC}}{2A} \geq \frac{-B - \sqrt{B^2 - 4AC}}{2A}$$

and thus we set the first contact time t_1 as

$$t_1 = \frac{-B + \sqrt{B^2 - 4AC}}{2A}. \tag{4}$$

It is because the contact occurs while a negative parabolic function for the ball runs downward. At this time the contact point is obtained by

$$\boldsymbol{p}_{t_1} := \boldsymbol{r}(t_1) = \ < v_{x_0}t_1 + p_{x_0}, \quad v_{y_0}t_1 + p_{y_0}, \quad -\frac{1}{2}\boldsymbol{g}t_1^2 + v_{z_0}t_1 + p_{z_0} > \tag{5}$$

and the velocity vector is

$$\boldsymbol{v}_{t_1} := \boldsymbol{r}'(t_1) = \ < v_{x_0}, \quad v_{y_0}, \quad -gt_1 + v_{z_0} > \tag{6}$$

Then the instant speed of the ball at the contact point is

$$\|\boldsymbol{v}_{t_1}\| = \sqrt{v_{x_0}^2 + v_{y_0}^2 + (-\boldsymbol{g}t_1 + v_{z_0})^2} \tag{7}$$

Step 3: Tangent plane and normal vector
In order to find the normal vector at the contact position $\boldsymbol{p}_{t_1}$, we should find the tangent plane at the point. In the rectangular $x - y - z$ coordinate system the general function representing a plane passing through a point (x_1, y_1, z_1) is written as

$$L(x, y): z = m_x(x - x_1) + m_y(y - y_1) + z_1$$

as a function of x and y in [15]. If y value is fixed, (and then $m_y(y - y_1) + z_1 = constant$), the tangent plane must be a tangent line to the 2 dimensional curve and thus the coefficient m_x of x should be the slope of tangent line, i.e. the derivative $f(x, y)$ with respect to x at (x_1, y_1). Thus

$$m_x = \frac{\partial f}{\partial x}(x, y)|_{(x_1, y_1)} = f_x(x_1, y_1)$$

Similarly, we have

$$m_y = \frac{\partial f}{\partial y}(x, y)|_{(x_1, y_1)} = f_y(x_1, y_1)$$

It says that the tangent plane passing through the point $\boldsymbol{p}_{t_1}$ to the parabolic wall $f(x, y) = ax^2 + by^2$ is written as

$$z = f_x(\boldsymbol{p}_{t_1})(x - \boldsymbol{p}_{t_1}(1)) + f_y(\boldsymbol{p}_{t_1})(y - \boldsymbol{p}_{t_1}(2)) + \boldsymbol{p}_{t_1}(3)$$

That is,

$$z - \left(-\tfrac{1}{2}gt_1^2 + v_{z_0}t_1 + p_{z_0}\right) - 2a\left(v_{x_0}t_1 + p_{x_0}\right)\left(x - \left(v_{x_0}t_1 + p_{x_0}\right)\right) - 2b\left(v_{y_0}t_1 + p_{y_0}\right)\left(y - \left(v_{y_0}t_1 + p_{y_0}\right)\right)$$
$$= 0$$

As an equivalent equation, it is written by

$$-2a\left(v_{x_0}t_1 + p_{x_0}\right)x - 2b\left(v_{y_0}t_1 + p_{y_0}\right)y + z = C_p$$

where

$$C_p := \left(-\tfrac{1}{2}gt_1^2 + v_{z_0}t_1 + p_{z_0}\right) - 2a\left(v_{x_0}t_1 + p_{x_0}\right)\left(v_{x_0}t_1 + p_{x_0}\right) - 2b\left(v_{y_0}t_1 + p_{y_0}\right)\left(v_{y_0}t_1 + p_{y_0}\right).$$

We have its normal vector $\boldsymbol{n}$ is

$$\boldsymbol{n} := < -2a\left(v_{x_0}t_1 + p_{x_0}\right), \quad -2b\left(v_{y_0}t_1 + p_{y_0}\right), \quad 1 > \tag{8}$$

One the other hand, the ball has a velocity vector in Eq. (6) at the contact moment and it is the tangent vector of the trajectory curve.
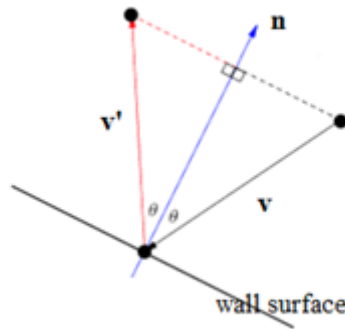

Figure 1. Diagram for reflection vector

Step 4: Reflection vector

To follow the assumption of Law of Reflection that the angle of incident light is the same as the angle of the reflected, we assume that the ball is not spinning and the wall is not deformable. Such an assumption states that the ball is reflected out with the same input angle after the impact. See **Error! Reference source not found.**.

Let $\boldsymbol{v}$ be the input vector and $\boldsymbol{n}$ the normal vector to the wall surface. To calculate the output vector $\boldsymbol{v}'$, refer to the figure.

The projection vector of **v** onto the normal vector **n** is

$$\|\boldsymbol{v}\|\cos\theta\,\frac{\boldsymbol{n}}{\|\boldsymbol{n}\|}$$

which is the half vector of $\boldsymbol{v} + \boldsymbol{v}'$ when $\boldsymbol{v}'$ has the same magnitude of **v**. (We look at an isosceles triangle.) That is,

$$\boldsymbol{v} + \boldsymbol{v}' = 2\|\boldsymbol{v}\|\cos\theta\,\frac{\boldsymbol{n}}{\|\boldsymbol{n}\|}. \tag{9}$$

Since the dot product $\boldsymbol{v}\cdot\boldsymbol{n} = \|\boldsymbol{v}\|\|\boldsymbol{n}\|\cos\theta$,

$$\frac{\boldsymbol{v}\cdot\boldsymbol{n}}{\|\boldsymbol{n}\|} = \|\boldsymbol{v}\|\cos\theta. \tag{10}$$

Substituting Eq. (10) into Eq. (9) and solving Eq. (9) for $\boldsymbol{v}'$, we have

$$v' = 2\frac{v \cdot n}{\|n\|^2}n - v$$

Applying this formula to the Eqs (6) and (8),

$$v' = \; < -2a(v_{x_0}t_1 + p_{x_0})\gamma - v_{x_0}, -2b(v_{y_0}t_1 + p_{y_0})\gamma - v_{y_0}, \gamma + gt_1 - v_{z_0} >$$

where $\gamma := \dfrac{-2(2av_{x_0}(v_{x_0}t_1+p_{x_0})+2bv_{y_0}(v_{y_0}t_1+p_{y_0})-gt_1-v_{z_0})}{\left(2a(v_{x_0}t_1+p_{x_0})\right)^2 + \left(2b(v_{y_0}t_1+p_{y_0})\right)^2 + 1}$. Thus the unit reflection vector $u := \dfrac{v'}{\|v'\|}$ becomes

$$u := \frac{< -2a(v_{x_0}t_1 + p_{x_0})\gamma - v_{x_0}, \; -2b(v_{y_0}t_1 + p_{y_0})\gamma - v_{y_0}, \; \gamma + gt_1 - v_{z_0} >}{\left(2a(v_{x_0}t_1 + p_{x_0})\gamma + v_{x_0}\right)^2 + \left(2b(v_{y_0}t_1 + p_{y_0})\gamma + v_{y_0}\right)^2 + \left(\gamma + gt_1 - v_{z_0}\right)^2} \tag{11}$$

Step 5: Rebound velocity

Through the STEP 5 we calculated the rebound direction of the ball after the impact. The rebound speed of the ball relies on the coefficients of restitution (COR) of the ball and the wall. The COR is the ratio of the final to initial relative velocity between two objects after they collide and it is related to (relative) kinetic energy. In the case of a one-dimensional collision involving two objects, object A and object B, the COR is given by

$$COR = \frac{v_{f_b} - v_{f_a}}{v_{I_b} - v_{I_a}}$$

where v_{f_a} and v_{f_b} are the final velocity of objects A, and B, respectively after impact and v_{I_a} and v_{I_b} are the initial velocity of objects A, and B, respectively before impact. Considering the kinetic energy in the three dimensional space, each velocity is replace as the speeds of objects. Moreover, assuming the wall to be always fixed (i.e. $v_{I_a} = v_{f_a} = 0$) and perfectly elastic (COR = 1) for collision (i.e. no kinetic energy is dissipated, and the rebound objects preserves the same relative speed), we have

$$The \; Ball \; Speed \; after \; impact = \alpha \cdot The \; ball \; speed \; before \; contact$$

where α is the COR of the ball. Then the speed of rebound ball becomes

$$\|v'_{t_1}\| = \alpha\sqrt{v_{x_0}^2 + v_{y_0}^2 + \left(-gt_1 + v_{z_0}\right)^2}$$

in the direction of u in Eq. (11) and thus the velocity of the rebound ball is

$$v'_{t_1}$$
$$:= \frac{\alpha\sqrt{v_{x_0}^2 + v_{y_0}^2 + \left(-gt_1 + v_{z_0}\right)^2} < -2a(v_{x_0}t_1 + p_{x_0})\gamma - v_{x_0}, \; 2b(v_{y_0}t_1 + p_{y_0})\gamma - v_{y_0}, \gamma + gt_1 - v_z}{\left(2a(v_{x_0}t_1 + p_{x_0})\gamma + v_{x_0}\right)^2 + \left(2b(v_{y_0}t_1 + p_{y_0})\gamma + v_{y_0}\right)^2 + \left(\gamma + gt_1 - v_{z_0}\right)^2} \tag{12}$$

Step 6: Piecewise trajectory function and Returning to Step 1

Now we obtained the trajectory function $r(t)$ in Eq. (1) and it runs from $t_0 = 0$ to t_1 in Eq. (4). After the first impact, then the ball continues the free fall movement which creates the sequential piece of the trajectory function $r(t)$ for t. The calculation processes are repeated in the exactly same manner from the Step 1 through Step 6 just with replacement of the initial position vector $< p_{x_0}, p_{y_0}, p_{z_0} >$ with $r(t_1) = p_{t_1}$ in Eq. (5) and the initial velocity $< v_{x_0}, v_{y_0}, v_{z_0} >$ with v'_{t_1} at Eq. (12). These steps are then terminated when the ball stops theoretically, i.e. $t_1 = 0$.

IV. Simulation in MATLAB

In order to simulate the movement of the ball in MATLAB software, **w**e consider the following conditions:

- Initial position of the ball: $p0 = [-1; 4; 80]$;
- Initial velocity of the ball: $v0 = [2; -2; 10]$;
- The parabolic wall: $z = x^2 + y^2$;

- COR of the ball: 0.35, 0.55, 0.85;
- The tolerance: $t = 0.01$ (The loop will stop when the jumping duration is less than 0.01 second)
  Then Figure 1 shows the results according to CORs of the ball. Each figure does not simulate rolling motion after stop jumping.
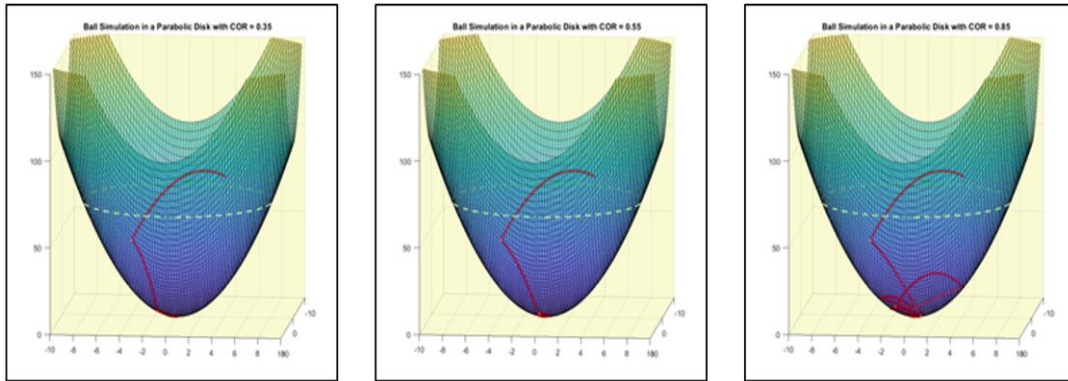
Figure 1. Ball simulation results in a parabolic disk with a) CORb = 0.35, b) CORb = 0.55, c) CORb = 0.85

With $COR_b = 0.35$ it makes a total 8 jumps and the total travel distance is 93.6334 m before it rolls. With $COR_b = 0.55$ the ball makes 14 jumps of the total travel distance 104.3347 m. When we set $COR_b = 0.85$, the total 45 jumps are made while moving 198.4341 m.

In addition, the simulation figure is translated into animation video using the MATLAB built-in functions

```
curve = animatedline('Linewidth',2);
set(gca,'Color',[250/255 250/255 210/255], 'XLim',[-10 10], 'YLim', [-10 10], 'ZLim', [0 150]);
view(96, 9);
holdon;

        fori=1:length(xplot)
addpoints(curve, xplot(i), yplot(i), zplot(i));
head = scatter3(xplot(i), yplot(i), zplot(i), 'filled');
                drawnow

                F = getframe(gcf);
                delete(head);
        end
        holdoff
```

V. Miscellaneous
From the moment when the ball stops jumping in the parabolic wall, it starts rolling along the wall because of its positive z values (height) without no external forces, such as rolling resistance acting on the ball. This rolling resistance includes rolling drag, the friction, etc. Then the ball is keep rolling up and down perpetually due to the principle of the conservation of mechanical energy

$$Total\ mechanical\ energy = \frac{1}{2}mv^2 + mgh = constant$$

while exchanging between kinetic and potential energy of a ball (Chapter 8 in [16]). For the 2-dimensional simulation, see [17].
The study of such motions can be written in differential equations as

$$\begin{cases} \dfrac{d^2}{dt^2}x(t) = 0 \\ \dfrac{d^2}{dt^2}y(t) = 0 \\ \dfrac{d^2}{dt^2}z(t) = -g \end{cases}$$

with the initial condition $[x(0),\ y(0),\ z(0)] = [p_{x_0},\ p_{y_0},\ p_{z_0}]$ and $[x'(0),\ y'(0),\ z'(0)] = [v_{x_0},\ v_{y_0},\ v_{z_0}]$. Then this second order initial value problem can be numerically solved using ODE solvers. In this case, the constraint of contacting a parabolic wall is treated by an event function. We refer a book [18].

VI. Conclusion and further discussion

We analyzed the motion of a ball using mathematics when it is thrown in the air within a parabolic wall and simulated in MATLAB. With the assumption that the wall is perfectly inelastic (i.e. COR = 0), the increase of COR of the ball increases the number of jumps inside and so lengthens the total travel distance. Once the ball stops jumping, it starts perpetually rolling like a swing of a pendulum due to the principle of energy conservation.

Furthermore, our research results can be applied to problems in the more interesting real-life environment while considering other external forces. For example, we may simulate the motion of a basketball dancing on the rim of the net or motion of a tennis ball thrown into a trash can.

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